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ON A CERTAIN CLASS OF RATIONAL RULED SURFACES.

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1. INTRODUCTION.

As is well known, ruled surfaces may be generated or defined in a number of ways. There exists, for example, a one-to-one correspondence between ruled surfaces, or scrolls as Cayley calls them, and a certain class of partial differential equations, so that the theories of the two classes are abstractly equivalent.

A much favored method, especially in descriptive geometry, consists in considering ruled surfaces as continuous sets of straight lines (generatrices), which intersect three fixed curves, (the directrices), simultaneously. If these are algebraic curves of orders l, m, n , with no points in common, the ruled surface which they define is, in general, of order $2lmn$.

Frequently ruled surfaces are also defined as systems of elements, either common to two rectilinear congruences, or to three rectilinear complexes.

Of great importance for the following investigation is the definition of ruled surfaces as systems of lines which join corresponding points of an (α, β) -correspondence between the points of two algebraic curves C_m and C_n of orders m and n . If the curves are plane, and if to a point of C_m correspond α points of C_n , and to a point of C_n β points of C_m , then the order of the surface is, in general, $\alpha m + \beta n$.

Finally there is the cinematic method in which ruled surfaces are generated by the continuous movement of the generatrix according to some definite cinematical law. In this connection the description of the hyperboloid of rotation of one sheet by a straight line rotating about a fixed axis is well known. The literature seems to contain but little about this method of generating ruled surfaces.* A number of treatises on differ-

In this paper the results of an investigation of a rather extended class of ruled surfaces are presented, which are defined cinematically. The class is

* E. M. Blake, "Two Plane Movements Generating Quartic Scrolls," *Transactions of the American Mathematical Society*, Vol. 1, pp. 421-429 (1900).

ential geometry contain chapters on cinematically generated surfaces.†

† Darboux, "Leçons sur la théorie générale des surfaces," Vol. 1, 2d ed., pp. 127-150 (1914).

Eisenhart, "A Treatise on the Differential Geometry of Curves and Surfaces," pp. 146-148 (1909).

obtained as follows: Given a directrix circle C_2 and a directrix line C_1 , which passes through the center of C_2 and is at right angles to the plane of C_2 . The generatrix g moves so that a fixed point M of g moves along C_2 uniformly, while g in every position passes through C_1 . The plane e through C_1 in which g lies evidently rotates about C_1 with the same velocity $k\theta$ as M . In this plane e , the generatrix g rotates at the same time with uniform velocity $\mu k\theta$ about M . It will be shown that g generates a rational algebraic surface of the class when $\mu = p/q$ is a rational fraction, and that this class thus described is equivalent to a class of ruled surfaces obtained by means of an (α, β) -correspondence between C_1 and C_2 , in which α and β will be determined hereafter.

Among the most important references bearing more closely upon the subject may be mentioned papers by Cremona,* Armamente,† Cayley,‡ Schwarz,§ Noether,|| Clebsch,¶ Picard.**

2. PARAMETRIC EQUATIONS AND ORDER OF THE SURFACE.

Let C_1 coincide with the z -axis, so that C_2 lies in the xy -plane, and g any position of the generatrix, P any point on g , P' its projection upon the xy -plane, and M the intersection of g with C_2 . By ρ denote the radius vector OP' , whose direction forms an angle θ with the positive part of the x -axis. In the plane e through C_1 , in which g lies, g is determined by the angle ψ which g makes with the positive direction of the perpendicular through M to the xy -plane, ψ being measured in the clock-wise sense. By the angles θ and ψ the position of g is perfectly determined. Assuming $\psi = 0$, when $\theta = 0$, and imposing upon ψ the condition $\psi = p/q \cdot \theta$, where p and q are relatively prime integers, the surface of the class, characterized by two definite values of p and q , may be represented parametrically by

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = (\rho - a) \cot \frac{p}{q} \theta, \quad (1)$$

* "Rappresentazione di una classe di superficie gobbe sopra un piano, e determinazione delle loro curve assintotiche," *Annali di Matematica*, Ser. II, Vol. 1, pp. 248–258 (1867).

† "Intorno alla rappresentazione della superficie gobbe di genere $p = 0$," *Annali di Matematica*, Ser. II, Vol. 4, pp. 50–72 (1870).

‡ "On Certain Skew Surfaces, otherwise Scrolls," *Transactions of the Cambridge Philosophical Society*, Vol. XI, Part II, pp. 277–289 (1869).

§ "Ueber die geradlinigen Flächen fünften Grades," *Journal für reine und angewandte Mathematik*, Vol. 76, pp. 23–57 (1867).

|| "Ueber Flächen, welche Scharen rationaler Curven besitzen," *Mathematische Annalen*, Vol. 3, pp. 161–227 (1871).

¶ "Ueber die geradlinigen Flächen vom Geschlechte $p = 0$," *Mathematische Annalen* Vol. 5, pp. 1–26 (1872).

** "Sur les surfaces algébriques dont toutes les sections planes sont unicursales," *Journal für reine und angewandte Mathematik*, Vol. 100, pp. 71–78 (1885).

in which a is the radius of C_2 , ρ and θ are the parametric coördinates of a point of the surface. Since

$$\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{1 - \tan^2 \left(q \cdot \frac{\theta}{2q} \right)}{1 + \tan^2 \left(q \cdot \frac{\theta}{2q} \right)},$$

$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{2 \cdot \tan \left(q \cdot \frac{\theta}{2q} \right)}{1 + \tan^2 \left(2q \frac{\theta}{q} \right)},$$

$$\cot \frac{p}{q} \theta = 1/\tan \left(2p \cdot \frac{\theta}{2q} \right),$$

may all be expressed as rational functions of the parameter $\tan \frac{\theta}{2q} = t$, x, y, z become rational functions of the parameters ρ and t , so that (1) defines a rational ruled surface.

To determine the order of the surface, the number of points of intersection of the line

$$a_1x = b_1y + c_1z + d_1 = 0, \quad a_2x + b_2y + c_2z + d_2 = 0 \quad (2)$$

with the surface must be found. Substituting in (2) for x, y, z their expressions given by (1), and eliminating ρ , the equation

$$\{(a_1d_2 - a_2d_1) \cos \theta + (b_1d_2 - b_2d_1) \sin \theta\} \tan \frac{p}{q} \theta$$

$$- (a_1c_2 - a_2c_1) \cos \theta - (b_1c_2 - b_2c_1) \sin \theta + c_1d_2 - c_2d_1 = 0 \quad (3)$$

is obtained. Making use of the identities

$$1 + \tan^2 m\theta = \cos^{2m} \theta (1 + \tan^2 \theta)^m / \cos^2 m\theta, \quad (4)$$

$$\sin m\alpha = \binom{m}{1} \cos^{m-1} \alpha \cdot \sin \alpha - \binom{m}{3} \cos^{m-3} \alpha \cdot \sin^3 \alpha + \dots, \quad (5)$$

$$\cos m\alpha = \binom{m}{0} \cos^m \alpha - \binom{m}{2} \cos^{m-2} \alpha \cdot \sin^2 \alpha + \dots, \quad (6)$$

and putting $\frac{\theta}{2q} = \alpha$, $\tan \alpha = t$,

$$\frac{\sin q \alpha}{\cos^q \alpha} = \binom{q}{1} \tan \alpha - \binom{q}{3} \tan^3 \alpha + \dots \pm \tan^{-q} \alpha = \phi^q(t),$$

$$\frac{\cos q \alpha}{\cos^q \alpha} = \binom{q}{0} - \binom{q}{2} \tan^2 \alpha + \dots \pm \binom{q}{q-1} \tan^{q-1} \alpha = \psi^{q-1}(t),$$

where ϕ and ψ are rational integral functions of t of order q and $q - 1$, as indicated by the upper indices. In this manner we obtain for $\sin \theta$, $\cos \theta$,

$\tan \frac{p}{q} \theta$ the expressions,

$$\sin \theta = \frac{2\phi^q(t)\psi^{q-1}(t)}{(1+t^2)^q}, \quad \cos \theta = \frac{\psi^{2q-2}(t) - \phi^{2q}(t)}{(1+t^2)^q}, \quad \tan \frac{p}{q} \theta = \frac{2\phi_1(t) \cdot \psi_1(t)}{\psi_1^2(t) - \phi_1^2(t)}. \quad (7)$$

The numerator of the expression for $\tan (p/q)\theta$ is of degree $2p - 1$, the denominator of degree $2p$ in t . Substituting these expressions in (5), an equation of degree $2(p + q)$ in t is obtained, so that when q is odd, a line (2) cuts the surface in as many points. Hence, *when q is odd, the order of the surface is $2(p + q)$.*

When $q = 2s$ is even, so that in (5)

$$\tan \frac{p}{q} \theta = \tan \left(p \cdot \frac{\theta}{2s} \right),$$

then putting $\tan (\theta/2s) = t$, with p odd, there is, in analogy with (7),

$$\begin{aligned} \sin \theta &= \frac{2 \tan \left(s \cdot \frac{\theta}{2s} \right)}{1 - \tan^2 \left(s \cdot \frac{\theta}{2s} \right)} = \frac{2\phi^s(t) \cdot \psi^{s-1}(t)}{(1+t^2)^s}, \\ \cos \theta &= \frac{1 - \tan^2 \left(s \cdot \frac{\theta}{2s} \right)}{1 + \tan^2 \left(s \cdot \frac{\theta}{2s} \right)} = \frac{\psi^{2s-2}(t) - \phi^{2s}(t)}{\psi^{2s-2}(t) + \phi^{2s}(t)}, \\ \tan \frac{p}{q} \theta &= \tan \left(p \cdot \frac{\theta}{2s} \right) = \frac{\psi^p(t)}{\psi^{p-1}(t)}, \end{aligned}$$

in which the upper indices indicate again the degrees of the polynomials ϕ and ψ in t . Substituting these expressions in (3), an equation of degree $p + q$ in t is obtained, so that in this case the order of the surface is $p + q$. The results may be stated as

THEOREM 1. *The surface of the class is rational and of order $2(p + q)$ or $p + q$, according as q is odd or even.*

3. CARTESIAN AND HOMOGENEOUS EQUATIONS OF THE SURFACE.

Applying the formula

$$\tan rw = \frac{\binom{r}{1} \tan w - \binom{r}{3} \tan^3 w + \binom{r}{5} \tan^5 w - \dots}{\binom{r}{0} - \binom{r}{2} \tan^2 w + \binom{r}{4} \tan^4 w - \dots} \quad (8)$$

to the identity

$$\tan p\theta = \tan q \cdot \frac{p\theta}{q}, \quad (9)$$

we get

$$\begin{aligned} & \frac{\binom{p}{1} \tan \theta - \binom{p}{3} \tan^3 \theta + \binom{p}{5} \tan^5 \theta - \dots}{\binom{p}{0} - \binom{p}{2} \tan^2 \theta + \binom{p}{4} \tan^4 \theta - \binom{p}{6} \tan^6 \theta + \dots} \\ &= \frac{\binom{q}{1} \tan \frac{p\theta}{q} - \binom{q}{3} \tan^3 \frac{p\theta}{q} + \binom{q}{5} \tan^5 \frac{p\theta}{q} - \dots}{\binom{q}{0} - \binom{q}{2} \tan^2 \frac{p\theta}{q} + \binom{q}{4} \tan^4 \frac{p\theta}{q} - \binom{q}{6} \tan^6 \frac{p\theta}{q} + \dots}. \end{aligned} \quad (10)$$

From (1)

$$\tan \theta = \frac{y}{x}, \quad \tan \frac{p\theta}{q} = \frac{\rho - a}{z}.$$

Substituting this in (10), and assuming q odd, after some reduction, the cartesian equation of the surface is obtained in the form

$$\begin{aligned} & \frac{\binom{p}{1} x^{p-1} y - \binom{p}{3} x^{p-3} y^3 + \binom{p}{5} x^{p-5} y^5 - \dots}{\binom{p}{0} x^p - \binom{p}{2} x^{p-2} y^2 + \binom{p}{4} x^{p-4} y^4 - \dots} \\ &= \frac{\binom{q}{1} (\rho - a) z^{q-1} - \binom{q}{3} (\rho - a)^3 z^{q-3} + \dots}{\binom{q}{0} z^q - \binom{q}{2} (\rho - a)^2 z^{q-2} + \dots} \\ & \quad + (-1)^{(q+1)/2} (\rho - a)^{q-2} z^2 + (-1)^{(q-1)/2} (\rho - a)^q \\ & \quad + (-1)^{(q+1)/2} (\rho - a)^{q-3} z^3 + (-1)^{(q-1)/2} (\rho - a)^{q-1} z. \end{aligned} \quad (11)$$

The expansions of $(\rho - a)^k$ bring in terms of the form $c_1(x^2 + y^2)^m$, $c_2(x^2 + y^2)^n \sqrt{x^2 + y^2}$, where c_1 and c_2 are definite constants. Hence, to rationalize the equation, all terms with $\sqrt{x^2 + y^2}$ as a factor must be collected on one side, all other terms on the other side of the equation. On squaring, a rational equation of order $2(p + q)$ is obtained, which is in agreement with the result of the foregoing section.

Monge* has shown that the equation of a ruled surface whose generatrices pass through the z -axis, has the form

$$z = x \psi \left(\frac{y}{x} \right) + \pi \left(\frac{y}{x} \right), \quad (12)$$

* "Application de l'Analyse à la Géométrie," 5 ed. (1850), pp. 83-89.

in which ψ and π are arbitrary functions (differentiable). This result he gained by differential geometry. Equation (11) can be readily put in the form (12). The left-hand member of (11) is a rational function of (y/x) ; the right-hand member a rational function of $(\rho - a)/z$. Consequently $(\rho - a)/z$ is an algebraic function of (y/x) , say

$$(\rho - a)/z = R(y/x).$$

From this $\rho^2 = \{a + zR(y/x)\}^2$, or $x^2(1 + (y/x)^2) = \{a + zR(y/x)\}^2$. Extracting the square root, which we may assume positive, and solving for z , we have

$$z = x \frac{\sqrt{1 + (y/x)^2}}{R(y/x)} - \frac{a}{R(y/x)}, \quad (13)$$

so that Monge's proposition is verified for our class of surfaces.

Making use of the identities

$$\begin{aligned} \binom{r}{1} A^{r-1} B - \binom{r}{3} A^{r-3} B^3 + \binom{r}{5} A^{r-5} B^5 - \dots \\ = -\frac{i}{2} \{(A + iB)^r - (A - iB)^r\} \end{aligned} \quad (14)$$

$$\begin{aligned} \binom{r}{0} A^r - \binom{r}{2} A^{r-2} B^2 + \binom{r}{4} A^{r-4} B^4 - \dots \\ = \frac{1}{2} \{(A + iB)^r + (A - iB)^r\} \end{aligned} \quad (15)$$

and introducing the homogeneous isotropic coördinates $x_1 = x + iy$, $x_2 = x - iy$, $x_3 = as + iz$, $x_4 = as - iz$, where s denotes the variable making the system (x, y, z) homogeneous, equation (11) reduces to the symmetric irrational form

$$x_1 x_2 (x_1^{p/q} - x_2^{p/q})^2 - (x_3 x_1^{p/q} - x_4 x_2^{p/q})^2 = 0. \quad (16)$$

The intersection of this surface with the plane at infinity of the system x, y, z, s , is obtained by putting $s = 0$, so that $x_3 = iz$, $x_4 = -iz$. Applying the collineation $x_1/z = \xi$, $x_2/z = \eta$, the curve of intersection then assumes the form

$$\xi \eta (\xi^{p/q} - \eta^{p/q})^2 + (\xi^{p/q} + \eta^{p/q})^2 = 0.$$

where ξ and η are again cartesian coördinates. Introducing polar coördinates by putting $\xi = \rho e^{i\theta}$, $\eta = \rho e^{-i\theta}$, this reduces to the simple form

$$\rho = \tan \frac{p}{q} \theta. \quad (17)$$

From the cinematic definition of the surface it is apparent that this curve may be considered as the projection of the curve at infinity from the origin

upon the plane $z = 1$. That it is rational as proved before, is also apparent if we put $\theta/q = w$, then $\xi = \tan pw \cdot \cos qw$, $\eta = \tan pw \cdot \sin qw$ may be expressed rationally by the parameter $\tan(w/2)$.

To investigate the behavior of the curve at the isotropic points, we may write its equation in the rational projective form

$$\begin{aligned} & (x_1^p + x_2^p)^2 x_1 x_2 \left\{ \binom{q}{1} x_3^{q-1} - \binom{q}{3} x_1 x_2 x_3^{q-3} + \dots \right. \\ & \quad \pm \binom{q}{4} x_1^{(q-5)/2} x_2^{(q-5)/2} x_3^4 \mp \binom{q}{2} x_1^{(q-3)/2} x_2^{(q-3)/2} x_3^2 \pm x_1^{(q-1)/2} x_2^{(q-1)/2} \left. \right\}^2 \\ & \quad + (x_1^p - x_2^p)^2 x_3^2 \left\{ \binom{q}{0} x_3^{q-1} - \binom{q}{2} x_1 x_2 x_3^{q-3} + \dots \right. \\ & \quad \mp \binom{q}{3} x_1^{(q-3)/2} x_2^{(q-3)/2} x_3^2 \pm \binom{q}{1} x_1^{(q-1)/2} x_2^{(q-1)/2} \left. \right\}^2, \end{aligned}$$

in which the algebraic signs within the brackets are alternating throughout.

“Placing the curve on the analytic triangle,” the terms nearest the vertex $X_1(x_2 = 0, x_3 = 0)$, after a rather tedious calculation, are found in order

$$x_1^{2p+q} x_2^q + \binom{q}{1} x_1^{2p+q-1} x_2^{q-1} x_3^2 + \binom{q}{2} x_1^{2p+q-2} x_2^{q-2} x_3^4 + \dots + x_1^{2p} x_3^{2q} + \dots,$$

and lie on a straight line of the “analytic triangle.” Putting $x_1 = 1$. this may be written as

$$(x_2 + x_3^2)^q + \dots,$$

so that in the neighborhood of the isotropic point $(x - iy = 0, z = 0)$ the curve has the same singularity as the curve $(x_2 + x_3^2)^q = 0$, which represents a q -fold parabola. This point must therefore be counted as $q(q - 1)$ double points. As the equation is symmetric with respect to x_1 and x_2 , also the other isotropic point $(x_1 = 0, x_3 = 0)$, or $(x + iy = 0, z = 0)$ must be counted as $q(q - 1)$ double-points. The isotropic points must therefore be counted as $2q(q - 1)$ double points.

When $q = 2s$ is even, the equation of the surface, which may also be written in the form

$$x_1^p(x_4 + i\rho)^q - x_2^p(x_3 - i\rho)^q = 0,$$

or

$$\begin{aligned} & x_1^p \left[x_4^q - \binom{q}{2} x_4^{q-2} \rho^2 + \binom{q}{4} x_4^{q-4} \rho^4 - \dots \right] \\ & \quad - x_2^p \left[x_3^q - \binom{q}{2} x_3^{q-2} \rho^2 + \binom{q}{4} x_3^{q-4} \rho^4 - \dots \right] \\ & = -i \left\{ x_1^p \left[\binom{q}{1} x_4^{q-1} \rho - \binom{q}{3} x_4^{q-3} \rho^3 + \dots \right] \right. \\ & \quad \left. + x_2^p \left[\binom{q}{1} x_3^{q-1} \rho - \binom{q}{3} x_3^{q-3} \rho^3 + \dots \right] \right\}, \end{aligned}$$

may be rationalized as follows:

Squaring both sides and making use of the identity

$$\left\{ \binom{r}{0} A^r - \binom{r}{2} A^{r-2} B^2 + \binom{r}{4} A^{r-4} B^4 - \dots \right\}^2 \\ + \left\{ \binom{r}{1} A^{r-1} B - \binom{r}{3} A^{r-3} B^3 + \binom{r}{5} A^{r-5} B^5 - \dots \right\}^2 = (A^2 + B^2)^r,$$

we get

$$x_1^{2p}(x_4^2 + \rho^2)^q + x_2^{2p}(x_3^2 + \rho^2)^q \\ - 2x_1^p x_2^p \left\{ \left[x_3^q - \binom{q}{2} x_3^{q-2} \rho^2 + \dots \right] \left[x_4^q - \binom{q}{2} x_4^{q-2} \rho^2 + \dots \right] \right. \\ \left. - \left[\binom{q}{1} x_3^{q-1} \rho - \binom{q}{3} x_3^{q-3} \rho^3 + \dots \right] \left[\binom{q}{1} x_4^{q-1} \rho - \binom{q}{3} x_4^{q-3} \rho^3 + \dots \right] \right\} = 0. \quad (18)$$

The bracket-expression multiplying $2x_1^p x_2^p$ may be written in the form

$$\frac{1}{4} \left[\{(x_3 + i\rho)^q + (x_3 - i\rho)^q\} \{(x_4 + i\rho)^q + (x_4 - i\rho)^q\} \right. \\ \left. + \{(x_3 + i\rho)^q - (x_3 - i\rho)^q\} \{(x_4 + i\rho)^q - (x_4 - i\rho)^q\} \right] \\ = \frac{1}{2} \{(x_3 + i\rho)^q (x_4 + i\rho)^q + (x_3 - i\rho)^q (x_4 - i\rho)^q\}.$$

Then, subtracting $2x_1^p x_2^p (x_3^2 + \rho^2)^{q/2} (x_4^2 + \rho^2)^{q/2}$ from the first two terms of (18), and adding the same expression to the third term, the equation may be written in the form

$$\{x_1^p (x_4^2 + \rho^2)^{q/2} - x_2^p (x_3^2 + \rho^2)^{q/2}\}^2 \\ = x_1^p x_2^p \{(x_3 + i\rho)^{q/2} (x_4 + i\rho)^{q/2} - (x_3 - i\rho)^{q/2} (x_4 - i\rho)^{q/2}\}^2. \quad (19)$$

Now, since $q = 2s$, be expanding,

$$(x_3 + i\rho)^s (x_4 + i\rho)^s = x_3^s x_4^s + i \binom{s}{1} \rho (x_3^{s-1} x_4^s + x_3^s x_4^{s-1}) \\ - i\rho^3 \left[\binom{s}{1} \binom{s}{2} x_3^{s-1} x_4^{s-2} + \binom{s}{1} \binom{s}{2} x_3^{s-2} x_4^{s-1} + \binom{s}{3} x_3^s x_4^{s-3} + \binom{s}{3} x_3^{s-3} x_4^s \right] \\ + \dots$$

The expansion of $(x_3 - i\rho)^s (x_4 - i\rho)^s$ is obtained from the foregoing by replacing i by $-i$. so that in the bracket-expression, multiplying $x_1^p x_2^p$ in (19), all terms without the factor $\pm i$ cancel, leaving for it

$$2i \left\{ \binom{s}{1} \rho (x_3^{s-1} x_4^s + x_3^s x_4^{s-1}) - \rho^3 \left[\binom{s}{1} \binom{s}{2} (x_3^{s-1} x_4^{s-2} + x_3^{s-2} x_4^{s-1} \right. \right. \\ \left. \left. + \binom{s}{3} (x_3^s x_4^{s-3} + x_3^{s-3} x_4^s) \right] + \dots \right\}.$$

Taking out the factor ρ , and noting that $\rho^2 = x_1x_2$, (19) now assumes the form

$$\begin{aligned} \{x_1^p(x_4^2 + x_1x_2)^s - x_2^p(x_3^2 + x_1x_2)^s\}^2 &= -4(x_1x_2)^{p+1} \left\{ \binom{s}{1} (x_3^{s-1}x_4^s + x_3^sx_4^{s-1}) \right. \\ &\quad - x_1x_2 \left[\binom{s}{1} \binom{s}{2} (x_3^{s-1}x_4^{s-2} + x_3^{s-2}x_4^{s-1}) + \binom{s}{3} (x_3^sx_4^{s-3} \right. \\ &\quad \left. \left. + x_3^{s-3}x_4^s) \right] + \dots \right\}^2. \end{aligned}$$

Extracting the square root on both sides of the equation which, according to geometric tests in examples given hereafter must be taken with the positive sign, we finally get for the equation of the surface, when q is even,

$$\begin{aligned} x_1^p(x_4^2 + x_1x_2)^s - x_2^p(x_3^2 + x_1x_2)^s & \\ - 2i(x_1x_2)^{p+1/2} \left\{ \binom{s}{1} (x_3^{s-1}x_4^s + x_3^sx_4^{s-1}) - x_1x_2 \left[\binom{s}{1} \binom{s}{2} (x_3^{s-1}x_4^{s-2} + x_3^{s-2}x_4^{s-1}) \right. \right. \\ \left. \left. + \binom{s}{3} (x_3^sx_4^{s-3} + x_3^{s-3}x_4^s) \right] + \dots \right\} &= 0. \quad (20) \end{aligned}$$

As p is odd, $p+1$ is even, and $(p+1)/2$ an integer, so that equation (20) is of degree $p+2s = p+q$, and the surface consequently of order $p+q$, as was proved before in the parametric form. From this the cartesian form is easily obtained by substituting for x_1, x_2, x_3, x_4 their expressions in terms of x, y, z .

As in the case of q odd, the intersection of the surface with the plane at infinity can be placed upon the analytic triangle. By the same method is found that the isotropic points must be counted as $2s(s-1)$ double points of the infinite curve of intersection in case of q even.

4. (α, β) -CORRESPONDENCE BETWEEN C_1 AND C_2 DETERMINED BY THE GENERATRICES OF THE SURFACE.

When we put $z = 0$ in (11), the intersection of the surface with the xy -plane is obtained

$$\left\{ \binom{p}{0} x^p - \binom{p}{2} x^{p-2}y^2 + \binom{p}{4} x^{p-4}y^4 - \dots \right\} (\rho - a)^q = 0. \quad (21)$$

To make this equation rational, put the bracket-expression equal to V , then $V^{1/q}(\rho - a) = 0$, $V^{2/q}(x^2 + y^2) = a^2V^{2/q}$, and from this as the equation of the intersection

$$V^2(x^2 + y^2 - a^2)^q = 0, \quad (22)$$

which shows that the directrix-circle C_2 is a q -fold curve of the surface,

when q is odd. The rest of the intersection consists of the p double generatrices in the xy -plane defined by $V^2 = 0$. These double lines divide the full angle into $2p$ equal parts, as is seen by actual determination of the angles from $V \equiv \rho^p(e^{ip\theta} + e^{-ip\theta}) = 0$, or $e^{2ip\theta} = -1$. From this is found $\theta = (k\pi/p) - (\pi/4p)$, so that θ has p values incongruent to π , determined by $k = 1, 2, 3, \dots, p$. When q is odd, the midpoint M of the generatrix g describes C_2 q times, so that θ increases by $q \cdot 2\pi$ and $(p/q)\theta$ by $p \cdot 2\pi$. From this follows that g turns around p times in the plane e and sweeps $2p$ times through the z -axis, so that the z -axis or C_1 is a $2p$ fold line of the surface. To determine the positions of the generatrices in a plane e determined by an angle θ more definitely, let g_0 be the initial position of the generatrix determined by the angles θ and $\psi = (p/q)\theta$. As θ increases by π , e turns through an angle π and the generatrix moves to the position g_1 making an angle $(p/q)\theta + (p/q)\pi$ with the perpendicular to the xy -plane through the midpoint M_1 of g_1 . As $\theta + \pi$ increases again by π , the generatrix g_1 moves to the new position g_2 through M , determined by the angle $(p/q)\theta + (p/q)2\pi$. While θ increases from 0 to $q \cdot 2\pi$, the generatrix describes the entire surface and returns point for point to its initial position g_0 . Denoting the $2q$ positions of the generatrix by $g_0, g_1, g_2, \dots, g_{2q-1}$ the corresponding determining angles may be listed in the table:

| | Through M | | Through M_1 |
|------------|-----------------------------|------------|-----------------------------|
| g_0 | $p/q\theta$ | g_1 | $p/q(\theta + \pi)$ |
| g_2 | $p/q(\theta + 2\pi)$ | g_3 | $p/q(\theta + 3\pi)$ |
| g_4 | $p/q(\theta + 4\pi)$ | g_5 | $p/q(\theta + 5\pi)$ |
| \vdots | \vdots | \vdots | \vdots |
| g_{2q-2} | $p/q\{\theta + (2q-2)\pi\}$ | g_{2q-3} | $p/q\{\theta + (2q-3)\pi\}$ |
| g_{2q} | $p/q\{\theta + 2q\pi\}$ | g_{2q-1} | $p/q\{\theta + (2q-1)\pi\}$ |

Through each M and M_1 there are q generatrices which divide the full angle around M as well as around M_1 in the plane e into $2q$ equal parts. Through every point of C_2 there are q generatrices.

To determine the number of generatrices through a point of C_1 , let such a point be determined by $z = \text{constant}$, then, $a\rho = 0$, from (1)

$$\cot \frac{p}{q}\theta = -\frac{z}{a} = \text{constant}. \quad (23)$$

From this $\frac{p}{q}\theta - k\pi = -\arccot \frac{z}{a}$, and

$$\theta = \frac{kq}{p}\pi - \frac{q}{p}\arccot \frac{z}{a}, \quad \left(\arccot \frac{z}{a} \leq \frac{\pi}{2} \right) \quad (24)$$

for $\theta \leq \theta \leq q \cdot 2\pi$. This gives for θ $2p$ values, determined by $k = 1, 2, 3, \dots, 2p$, and incongruent $q \cdot 2\pi$, and consequently $2p$ generatrices through a point of the z -axis, or C_1 . These lie in pairs of symmetric lines in planes through C_1 , each pair being determined by two values k and k_1 , such that their difference $k_1 - k = p$.

From the fact, that to every point of C_1 correspond $2p$ points of C_2 , the points of intersection of the $2p$ generatrices through the point on C_1 ; and to every point of C_2 q points of C_1 , the intersections of the q generatrices through the point on the C_2 with C_1 , it is apparent that the generatrices cut out on C_1 and C_2 an (α, β) -correspondence of the specific form $(q, 2p)$.

To establish the algebraic form of this correspondence, we may represent C_2 parametrically by

$$x = a \frac{1 - \lambda^2}{1 + \lambda^2}, \quad y = a \frac{2\lambda}{1 + \lambda^2},$$

so that conversely $\lambda = y/(a + x)$. If we now put $\tan(\theta/2q) = t$, then for $\rho = 0$,

$$z = -a \cot \frac{p}{q} \theta = -a \cot 2p \cdot \frac{\theta}{2q} = F(t), \quad (25)$$

$$\lambda = \frac{\sin \theta}{1 + \cos \theta} = \frac{2 \tan q \cdot \frac{\theta}{2q}}{1 - \tan^2 q \cdot \frac{\theta}{2q}} = G(t), \quad (26)$$

both rational functions of t . In $F(t)$ the highest power of t is $2p$, in $G(t)$ it is q . The elimination of t between (25) and (26) leads to a polynomial relation between z and λ which is of degree q in z , and of degree $2p$ in λ , and of degree $2p + q$ in both, and which is of deficiency 0. It has the form

$$\lambda^{2p}(a_0 z^q + a_1 z^{q-1} + \dots) + \lambda^{2p-1}(b_0 z^q + b_1 z^{q-1} + \dots) + \dots + w_0 z^q + w_1 z^{q-1} + \dots = 0. \quad (27)$$

This, when considered as a curve in a (λ, z) -plane, being of deficiency 0, has $\frac{1}{2}(2p + q - 2)(2p + q - 1)$ double-points, or double roots. But the infinite points of the λ and z -axis, are in the same order q and $2p$ -fold points so that the number of finite double points is

$$\begin{aligned} \frac{1}{2}\{(2p + q - 2)(2p + q - 1) - 2p(2p - 1) - q(q - 1)\} \\ = 2pq - 2p - q + 1. \end{aligned} \quad (28)$$

But there are no real double generatrices which are simultaneously double lines of the system of $2p$ generatrices through a point of the z -axis and the q generatrices through a point of C_2 . The surface has therefore a number of imaginary double-generatrices as given by (28). Hence

THEOREM 2. When q is odd, the generatrices of the surface cut C_1 and C_2 in two point sets which are in a $(q, 2p)$ -correspondence. C_1 and C_2 are $2p$ -fold and q -fold curves of the surface. The surface has moreover p real and $2pq - 2p - q + 1$ imaginary double generatrices.

When $q = 2s$ is even, $\psi = (p/2s)\theta$ increases from 0 to $p\pi$, when θ increases from 0 to $s \cdot 2\pi$. The midpoint M of the generatrix g turns s times about the z -axis, and g turns p times around M in the plane e . From this follows that C_1 and C_2 are respectively p - and s -fold lines of the surface. When θ increases from θ_0 to $\theta_0 + s \cdot 2\pi$, the generatrix g moves from the initial position g_0 to the position g_{2s} which coincides with g_0 , but the segments into which M divides g_0 are interchanged on g_{2s} . In other words g describes a closed unifacial ruled surface.

From a point of C_1 there are p generatrices cutting C_2 . This appears from the formula

$$\theta = \frac{kq}{p}\pi - \frac{q}{p} \operatorname{arccot} \frac{z}{a} \quad \left(\operatorname{arccot} \frac{z}{a} \leqq \frac{\pi}{2} \right),$$

with $0 \leqq \theta \leqq s \cdot 2\pi$, which determines p values for θ by putting $k = 1, 2, 3, \dots, p$. For two distinct values k and k_1 of this set we have $\theta_1 - \theta = [(k_1 - k)/p]q\pi$. As p is odd, q even, and $k_1 - k < p$, and p and q are relatively prime, $\theta_1 - \theta$ can never be an odd multiple of π . Hence no two of the generatrices through a point of C_1 can lie in the same plane e ; hence the surface has no real double generatrices.

Through every point of C_1 there are p generatrices cutting C_2 in as many points; through every point of C_2 there are s generatrices cutting C_1 in the same number of points. Hence

THEOREM 3. When $q = 2s$ is even the, generatrices of the surface cut C_1 and C_2 in two point sets which are in a rational (s, p) -correspondence. C_1 and C_2 are respectively p - and s -fold curves of the surface. The surface has no real, but has $ps - p - s + 1$ imaginary double generatrices.

5. DOUBLE CURVES OF THE SURFACE WHEN q IS ODD.

The $2q$ generatrices in a plane e intersect in q^2 points D_{ik} of the double curve of the surface. Each pair of indices ik consists throughout of an odd and an even number. These q^2 couples may be arranged into $(q + 1)/2$ cyclic groups of which $(q - 1)/2$ are of order $2q$, and 1 is of order q . Those of order $2q$ are

| | | | | |
|-------------------|------------------|------------------|-----|------------------------|
| 01 | 03 | 05 | ... | $0 \cdot q - 2$ |
| 12 | 14 | 16 | ... | $1 \cdot q - 1$ |
| 23 | 25 | 27 | ... | $2 \cdot q$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $2q - 1 \cdot 2q$ | $2q - 1 \cdot 2$ | $2q - 1 \cdot 4$ | ... | $2q - 1 \cdot q - 3$, |

that of order q is

$$\begin{array}{c} 0 \\ 1 \cdot q + 1 \\ 2 \cdot q + 2 \\ \vdots \\ q - 1 \cdot 2q - 1. \end{array}$$

In this group the generatrices g_0 and g_q , intersecting at D_{0q} , are determined by the angles $(p/q)\theta$ and $(p/q)(\theta + q\pi) = (p/q)\theta + p\pi$, which shows that when θ varies, the corresponding generatrices g_0 and g_q always intersect in a point of C_1 , i.e., the intersections D of the group of order q all lie on C_1 . On the other hand the intersections of each group of order $2q$ are cyclically permuted when θ increases by $q \cdot 2\pi$, so that the points $D_{01}, D_{03}, D_{05}, \dots, D_{0 \cdot q-1}$ describe $(q - 1)/2$ twisted curves which together form the double curve of the ruled surface.

To find the equation of any of these double curves, for example the one described by the point starting from $D_{0 \cdot 2k-1}$, where k may have any value between 1 and $(q - 1)/2$, let x, y, z be the cartesian coördinates of $D_{0 \cdot 2k-1}$, and ρ, θ the polar coördinates of x, y . The equations of g_0 and $g_{0 \cdot 2k-1}$ in their planes (ρ, z) are

$$z = (\rho - a) \cot \frac{p}{q} \theta, \quad z = -(\rho - a) \cot \frac{p}{q} \{\theta + (2k - 1)\pi\}.$$

From this

$$\rho = \frac{a \sin \frac{p}{q} (2k - 1)\pi}{\sin \frac{p}{q} \{2\theta + (2k - 1)\pi\}}, \quad (29)$$

which is the polar equation of the projection of the double curve upon the xy -plane. As $x = \rho \cos \theta, y = \rho \sin \theta$, the parametric (θ) equations of the double curve are

$$\begin{aligned} x &= \frac{a \sin \frac{p}{q} (2k - 1)\pi \cdot \cos \theta}{\sin \frac{p}{q} \{2\theta + (2k - 1)\pi\}}, \\ y &= \frac{a \sin \frac{p}{q} (2k - 1)\pi \cdot \sin \theta}{\sin \frac{p}{q} \{2\theta + (2k - 1)\pi\}}, \\ z &= -a \frac{\cos \frac{p}{q} \{2\theta + (2k - 1)\pi\} + \cos \frac{p}{q} (2k - 1)\pi}{\sin \frac{p}{q} \{2\theta + (2k - 1)\pi\}}. \end{aligned} \quad (30)$$

From this all double curves are obtained by putting successively $k = 1, 2, 3, \dots, (q-1)/2$.

Eliminating θ between (29) and the last equation of (30), the relation between ρ and z becomes

$$(\rho^2 - z^2 - a^2) \sin \frac{p}{q} (2k-1)\pi = 2\rho z \cos \frac{p}{q} (2k-1)\pi, \quad (31)$$

which shows that the $2q$ intersections of the generatrices of a group in any plane e through the z -axis lie on two hyperbolas, which are symmetrical with respect to the z -axis, since to every increase of θ by π corresponds a rotation of the hyperbola (31) about the z -axis through an angle π . Thus the points $D_{01}, D_{23}, D_{45}, \dots$ lie on (14), while $D_{12}, D_{34}, D_{56}, \dots$ lie on the reflexion of (14). The double curve belonging to the group lies therefore on a surface of revolution of order 4 with the equation

$$(x^2 + y^2 - z^2 - a^2)^2 \sin^2 \frac{p}{q} (2k-1)\pi - 4(x^2 + y^2)z^2 \cos^2 \frac{p}{q} (2k-1)\pi = 0, \quad (32)$$

with C_2 as a double curve.

The asymptotes of the hyperbola (31) are determined by the equation of their slope z/ρ ,

$$\left(\frac{z}{\rho}\right)^2 + 2\left(\frac{z}{\rho}\right) \cot \frac{p}{q} (2k-1)\pi - 1 = 0,$$

which shows that the product of the slopes of the asymptotes is -1 . (31) is therefore an equilateral hyperbola.

To find the generatrices of the surface with the same slopes, in other words the infinite points of the double curve described by $D_{0 \cdot 2k-1}$, the conditions must be satisfied:

$$(a) \quad \cot \frac{p}{q} \theta = \tan \frac{p(2k-1)\pi}{2q},$$

and from this

$$\theta = \frac{q(2l+1)}{2p} \pi - \frac{(2k-1)\pi}{2},$$

for all values $l = 0, 1, 2, \dots, 2p-1$. Two consecutive values of θ differ by the amount $(q/2p)\pi$. For every value of θ of this set there is a value $\theta' = \theta + q\pi$. For a definite value of l the angles are

$$\theta' = \frac{q\{2(l+p)+1\}}{2p} \pi - \frac{2k-1}{2} \pi \text{ and } \theta = \frac{q(2l+1)}{2p} \pi - \frac{2k-1}{2} \pi.$$

Hence there are $2p$ values of θ due to condition (1) for which the double curve has infinite points. The condition

$$(b) \quad \cot \frac{p}{q} \theta = - \cot \frac{p}{2q} (2k-1)\pi$$

is satisfied, when

$$\theta = \frac{ql\pi}{p} - \frac{2k-1}{2}\pi,$$

which again determines $2p$ values for θ , for $l = 1, 2, 3, \dots, 2p-1$. The values of θ determined by (a) and (b) are also obtained from the condition that in (29) $\rho = \infty$. This is the case for $(p/q)\{2\theta + (2k-1)\pi\} = m\pi$, or

$$\theta = \frac{mq\pi}{2p} - \frac{(2k-1)\pi}{2},$$

for $m = 1, 2, 3, \dots, 4p$, and $k \leq (q-1)/2$. For $k = (q+1)/2$ and $m = 1$, we get $\theta = 0$ and $\rho = a$. This corresponds to the one group of order q .

The order of the double curve may be determined by expressing sines and cosines in (30) by tangents. In the first place let $\theta + [(2k-1)/2]\pi = \phi$, or $\theta = -\{(2k-1)(\pi/2) - \phi\}$, so that $\sin \theta = (-1)^k \cos \phi$, $\cos \theta = (-1)^{k+1} \sin \phi$. Then (30) may be written in the form

$$\begin{aligned} x &= a \cdot \frac{(-1)^{k+1} \sin \frac{p}{q} (2k-1)\pi \cdot \sin \phi}{\sin \frac{p}{q} 2\phi}, \\ y &= a \cdot \frac{(-1)^k \sin \frac{p}{q} (2k-1)\pi \cos \phi}{\sin \frac{p}{q} 2\phi}, \\ z &= -a \cdot \frac{\cos \frac{p}{q} 2\phi + \cos \frac{p}{q} (2k-1)\pi}{\sin \frac{p}{q} 2\phi}. \end{aligned} \quad (33)$$

Expressing sines and cosines in terms of $\tan(\theta/2q) = t$ by means of identities as given in (9), and using similar notations by functions $\phi, \psi, \phi_1, \psi_1$, of t the parametric equations of the double curve become

$$\begin{aligned} x &= (-1)^{k+1} a \sin \frac{p}{q} (2k-1)\pi \frac{\phi^q(t) \psi^{q-1}(t) (1+t^2)^{2p-q}}{\phi^{2p}(t) \cdot \psi^{2p-1}(t)}, \\ y &= (-1)^k a \sin \frac{p}{q} (2k-1)\pi \frac{\{\psi^{2q-1}(t) - \phi^{2q}(t)\} (1+t^2)^{2p-q}}{2\phi^{2p}(t) \cdot \psi^{2p-1}(t)}, \\ z &= -a \frac{\psi^{4p-2}(t) - \phi^{4p}(t) + \cos \frac{q}{p} (2k-1)\pi \cdot (1+t^2)^{2p}}{2\phi^{2p}(t) \cdot \psi^{2p-1}(t)}, \end{aligned} \quad (34)$$

which are rational functions of t . The highest power of t in these functions is $4p$, or $2q$, according as $q \leq 2p$. Hence

THEOREM 4. *The order of each of the $(q - 1)/2$ double curves is $4p$ or $2q$ according as $q \leq 2p$, and when q is odd. They are rational and each lies on a surface of revolution of order 4 generated by the rotation of an equilateral hyperbola about the z -axis.*

As the surfaces of the class are rational, all plain sections are rational and must therefore have the maximum number of double points. We shall verify this number in case of q odd, and $q \leq 2p$, for the curve of intersection with the plane at infinity. Consider first a plane $z = \text{const.}$, cutting the z -axis in a point Z . A pair of generatrices g and g' in the same plane e and through the same point U (different from Z) of the z -axis cut the plane $z = \text{const.}$ in two points A and A' , which are collinear with Z . Now as θ increases, A and A' describe curves in central symmetry with Z , which will touch each other at Z , i.e., the form or tac-nod at Z . But Z must be counted as a $2p$ -fold point of the curve of intersection with the plane $z = \text{const.}$, since the z -axis is a $2p$ -fold line of the surface, and as it occurs p times that two branches of the curve form tac-nods, the point Z must be counted as

$$\frac{2p(2p - 1)}{2} + p = 2p^2$$

double points. As this is true for every point of the z -axis, the latter must be counted as $2p^2$ double lines. The surface has therefor $2p^2 + p$ real double lines (p double lines in the xy -plane). The double points of the infinite curve of intersection are therefore made up of those absorbed by the isotropic points, of the intersections with the real double lines, with the imaginary double-lines, and with the $(q - 1)/2$ double curves, each with $4p$ infinite points. These numbers are in the same order, and summed up:

$$\begin{aligned} & \{2q(q - 1)\} + \{2p^2 + p\} + \{2pq - 2p - q + 1\} \\ & + \left\{ \frac{q - 1}{2} \cdot 4p \right\} = \frac{\{2(p + q) - 1\}\{2(p + q) - 2\}}{2}, \end{aligned}$$

as required of a plain curve of order $2(p + q)$.

6. DOUBLE CURVES OF THE SURFACE WHEN g IS EVEN.

When $q = 2s$, there are two pencils, of s generatrices each, in every plane section through the z -axis, which intersect in s^2 points. When the plane e turns s times about the z -axis, starting from an initial position e_0 containing the initial position g_0 of the generatrix g , assuming in succession positions determined by the angles $\theta = 0, \pi, 2\pi, 3\pi, \dots$, g will occupy the positions $g_0, g_1, g_2, \dots, g_{2s-1}$. In this movement the intersection of the generatrices of even and odd indices in the same plane describe double curves of the surface. Every double curve, with the exception of one in

case when s is odd, cuts the initial plane e_0 in $2s$ points. When s is odd the number of intersections of the one curve with the plane is s , and as

$$s^2 = \frac{s-1}{2} \cdot 2s + s,$$

there are in this case $(s-1)/2$ double curves with $2s$ intersections, and one curve with s intersections with the plane e_0 ; i.e., altogether $(s+1)/2$ double curves. To each of these curves corresponds a cyclic group of points of intersection with the plane e_0 . Denoting the groups of order $2s$ by $D_1, D_2, D_3, \dots, D_{(s-1)/2}$ and the group of order s by $D_{(s-1)/2}$ the following table of these groups may be set up:

| D_1 | D_2 | D_3 | \dots | $D_{(s-1)/2}$ | $D_{(+1s)/2}$ |
|-----------------------|------------------|------------------|---------|----------------------|--------------------------|
| 01 | 03 | 05 | \dots | $0 \cdot (s-2)$ | $0 \cdot (s)$ |
| 12 | 14 | 16 | \dots | $1 \cdot (s-1)$ | $1 \cdot (s+1)$ |
| 23 | 25 | 27 | \dots | $2 \cdot (s+1)$ | $2 \cdot (s+2)$ |
| \vdots | \vdots | \vdots | \dots | \vdots | \vdots |
| $(2s-2) \cdot (2s-1)$ | $(2s-2) \cdot 1$ | $(2s-2) \cdot 3$ | \dots | $(2s-2) \cdot (s-4)$ | $(s-1) \cdot [2(s-1)+1]$ |
| $(2s-1) \cdot 0$ | $(2s-1) \cdot 2$ | $(2s-1) \cdot 4$ | \dots | $(2s-1) \cdot (s-3)$ | |

When $s = 2\sigma$ is even, then

$$s^2 = 4\sigma^2 = \sigma \cdot 4\sigma = \sigma \cdot 2s.$$

There are σ double curves, each with $2s$ intersections with the initial plane. The table of groups is in this case:

| D_1 | D_2 | D_3 | \dots | D_σ |
|-----------------------|------------------|------------------|---------|----------------------|
| 01 | 03 | 05 | \dots | $0 \cdot (s-1)$ |
| 12 | 14 | 16 | \dots | $1 \cdot (s)$ |
| 23 | 25 | 27 | \dots | $2 \cdot (s+1)$ |
| \vdots | \vdots | \vdots | \dots | \vdots |
| $(2s-2) \cdot (2s-1)$ | $(2s-2) \cdot 1$ | $(2s-2) \cdot 3$ | \dots | $(2s-2) \cdot (s-3)$ |
| $(2s-1) \cdot 0$ | $(2s-1) \cdot 2$ | $(2s-1) \cdot 4$ | \dots | $(2s-1) \cdot (s-2)$ |

The parametric equations of the double curves are obtained from those of (33) by putting $\tan(\phi/2s) = t$, so that in analogy with previous results

$$\sin \phi = \frac{2 \tan s \cdot \frac{\phi}{2s}}{1 + \tan^2 s \cdot \frac{\phi}{2s}} = \frac{2\phi^s(t) \cdot \psi^{s-1}(t)}{(1 + t^2)^s},$$

$$\cos \phi = \frac{1 - \tan^2 s \cdot \frac{\phi}{2s}}{1 + \tan^2 s \cdot \frac{\phi}{2s}} = \frac{\psi^{2(s-1)}(t) - \phi^{2s}(t)}{(1 + t^2)^s},$$

$$\cos \frac{p}{s} \phi = \frac{1 - \tan^2 p \cdot \frac{\phi}{2s}}{1 + \tan^2 p \cdot \frac{\phi}{2s}} = \frac{\phi_1^{2(p-1)}(t) - \psi_1^{2p}(t)}{(1 + t^2)^p},$$

and the parametric equations of the double curve are

$$\begin{aligned} x &= (-1)^{k+1} a \sin \frac{p}{2s} (2k-1)\pi \frac{\phi^s(t) \psi^{s-1}(t) (1+t^2)^{p-s}}{\phi_1^p(t) \cdot \psi_1^{p-1}(t)}, \\ y &= (-1)^k a \sin \frac{p}{2s} (2k-1)\pi \frac{\{\psi^{2(s-1)}(t) - \phi^{2s}(t)\} (1+t^2)^{p-s}}{2\phi_1^p(t) \cdot \psi_1^{p-1}(t)}, \quad (35) \\ z &= -a \frac{\phi_1^{2(p-1)}(t) - \psi_1^{2p}(t) + \cos \frac{p}{2s} (2k-1)\pi \cdot (1+t^2)^{p-s}}{2\phi_1^p(t) \cdot \psi_1^{p-1}(t)}. \end{aligned}$$

When $p > s$, or $q < 2p$, and $2k-1 \neq s$, then the highest power of t in these expressions is t^{2p} , so that in this case the double curves are of order $2p$. When $p < s$, then they are of order $2s = q$.

In case of the double curve of order s , which is obtained from (33) by putting $2k-1=s$, when s is odd, the coördinates may be expressed parametrically by $t = \tan(\phi/s)$ as rational functions of t , in which the highest power of t is t^p or t^s , according as $p > s$, or $p < s$, so that the double curve $[D(s+1)/2]$ is accordingly of order p or s , as appears from the parametric equations of the surface

$$\begin{aligned} x &= \pm \frac{(1+t^2)^{(p-s)/2} \left[\binom{s}{1} - \binom{s}{3} (1+t^2)t^2 + \dots \pm t^{s-1} \right]}{\binom{p}{1} - \binom{p}{3} (1+t^2)t^2 + \dots \pm t^{p-1}}, \\ y &= \pm \frac{(1+t^2)^{(p-s)/2} \left[1 - \binom{s}{2} t^2 + \dots \pm t^{s-1} \right]}{\binom{p}{1} t - \binom{p}{3} t^3 + \dots \pm t^p}, \quad (36) \\ z &= -a \cot \left(\frac{p}{s} \phi \right) = f(t), \end{aligned}$$

where the signs within the brackets and in the denominators are alternating throughout, and $f(t)$ denotes a rational function in which the highest power of t is t^p . Hence

THEOREM 5. When $q = 2s$ is even and s odd, there are $(s - 1)/2$ double curves of order $2p$ or q according as $p > s$ or $p < s$, and one double curve of order p or s , according as $p > s$ or $p < s$. When $s = 2\sigma$ is even, there are σ double curves of order $2p$ or q , according as $p > s$ or $p < s$.

7. APPLICABILITY AMONG THE RULED SURFACES OF THE CLASS.*

Consider first the case of an odd q . Let the generatrix g whose midpoint M moves along the directrix circle C_2 generate the surface determined by the numbers p and q . At every position of M draw in the xy -plane an external tangent circle C'_2 to C_2 with the radius $(m/n)a$. Through the center $0'$ of C'_2 draw a perpendicular z' to the plane of C'_2 , and let g in every position be associated with C'_2 and z' , just as it is associated with C_2 and z . Thus, the generatrices g associated with C'_2 , z' will generate a ruled surface F' whose directrix circle is C'_2 and whose directrix line is z' . To determine the nature of this surface, we must determine how many revolutions C'_2 has to make about the axes z' and z respectively, before the initial point M_0 on C'_2 , after a certain number of revolutions about the z' -axis, and the initial generatrix g_0 associated with F' , after rotating a certain number of times about M in the plane e' , return to the initial positions of M on C_2 , and g_0 associated with the given surface. This will be the case when a certain multiple ν of the circumference of C'_2 is equal to a certain multiple of q times the circumference of C_2 , i.e., $\nu \cdot 2(m/n)a\pi = \mu q \cdot 2a\pi$. This gives for μ/ν the ratio

$$\frac{\mu}{\nu} = \frac{m}{nq}.$$

Assuming m and n , also m and q as relative primes, we may put $\mu = m$, $\nu = nq$. From this is seen that the ruled surface F' is generated by a generatrix whose midpoint describes C'_2 nq times. The surface F' turns mq times about the z -axis. Moreover g associated with F' makes mp complete revolutions in the plane e' through the z' -axis. The order of the surface is therefore $2(mp + nq)$. Hence

THEOREM 6. Surfaces of the class are applicable to each other when their orders are $2(p + q)$ and $2(mp + nq)$, and their radii of C_2 respectively a and $(m/n)a$, when q is odd, p and q , m and n , and m and q are relative primes. This is still true when either $p = q = 1$, or $m = n = 1$, hence

Corollary 7. A surface of the class of order $2(p + q)$, q odd, is applicable upon a surface of the class of order 4.

Corollary 8. A surface of the class of order $2(p + q)$, q odd, is applicable to another surface of the same class and order.

* See Eisenhart, loc. cit., pp. 342-347.

When q is even, $q = 2s$, then we have under similar conditions

$$\frac{\mu}{\nu} = \frac{m}{ns},$$

and $mp + nq$ as the order of the surface. As m and p are odd, and q is even, $mp + nq$ is an odd number, so that we may state

THEOREM 9. *Surfaces of the class of odd order are applicable to each other when their orders are $p + q$ and $mp + nq$, and their radii a and $(m/n)a$, respectively, and when q is even and p and m are odd.*

As the surfaces of even and odd order are bifacial and unifacial respectively, we have

Corollary 10. *Bifacial and unifacial surfaces of the class are applicable to surfaces of the same type only.*

Similar results would be obtained by choosing for C'_2 an internal tangent circle of C_2 .

8. INTERSECTIONS OF THE SURFACES OF THE CLASS WITH A TORUS.

A torus whose circular axis coincides with the directrix circle C_2 of the surface cuts the surface in a composite curve of order $8(p + q)$ or $4(p + q)$, according as q is odd or even. If the radius of a meridian cross-section of the torus is b , then it follows easily that a part of this composite curve, generated by a point P on the generatrix g whose distance $PM = b$ from M is constant may be represented parametrically by

$$\begin{aligned} x &= \left(a + b \sin \frac{p}{q} \theta \right) \cos \theta, \\ y &= \left(a + b \sin \frac{p}{q} \theta \right) \sin \theta, \\ z &= b \cos \frac{p}{q} \theta. \end{aligned} \tag{37}$$

Putting again $\tan(\theta/2q) = t$, x, y, z may be expressed as rational functions of t , in which the highest power of t is $t^{2(p+q)}$, so that the order of the curve is $2(p + q)$. The polar equation of the projection of the curve upon the xy -plane is

$$\rho = a + b \sin \frac{p}{q} \theta, \tag{38}$$

which represents a socalled cyclo-harmonic curve,* which, in general, is of order $2(p + q)$, and is rational.

* R. E. Moritz, "On the Construction of Certain Curves Given in Polar Coördinates, *The American Mathematical Monthly*, Vol. XXIV, pp. 213–220 (1917).

When $a = 0$, and one of the integers p, q is even, the other odd, the curve is still of order $2(p + q)$; but when both are odd, then, as is well known,* the curve is of order $p + q$.

Expanding the identity $\sin p\theta = \sin q \cdot (p\theta/q)$, and substituting

$$\sin \frac{p}{q} = \frac{\rho - a}{b}, \quad \sin \theta = \frac{y}{\rho}, \quad \cos \theta = \frac{x}{\rho}, \quad \cos \frac{p}{q} \theta = \frac{\sqrt{b^2 - (\rho - a)^2}}{b},$$

the cartesian equation of the curve (38) may be written in the form

$$\begin{aligned} b^q & \left\{ \binom{p}{1} x^{p-1} y - \binom{p}{3} x^{p-3} y^3 + \binom{p}{5} x^{p-5} y^5 - \dots \right\} \\ &= \rho^p \left\{ \binom{q}{1} [b^2 - (\rho - a)^2]^{(q-1)/2} (\rho - a) \right. \\ &\quad \left. - \binom{q}{3} [b^2 - (\rho - a)^2]^{(q-3)/2} (\rho - a)^3 + \dots \right\}. \end{aligned} \quad (39)$$

9. EXAMPLES.

A. Bifacial Surfaces.

1. *Quartic*,† $p = 1, q = 1$.

$$(x^2 + y^2)x^2 - (ax + yz)^2 = 0.$$

C_2 is a single curve, C_1 must be counted as two double lines. The y -axis is the only double line in the xy -plane.

2. *Sextic*, $p = 2, q = 1$.

$$(x^2 + y^2)(x^2 - y^2)^2 - \{2xyz - a(x^2 - y^2)\}^2 = 0.$$

C_2 is a single curve, C_1 must be counted as eight double lines, and there are two double lines in the xy -plane with the equations $x \pm y = 0$.

B. Unifacial Surfaces.

3. *Cubic*, $p = 1, q = 2$

$$\{(x - a)^2 - (z - y)^2\}y + 2x(x - a)(z - y) = 0,$$

or in parametric form

$$x = \rho \frac{1 - t^2}{1 + t^2}, \quad y = \rho \frac{2t}{1 + t^2}, \quad z = \frac{\rho - a}{t}.$$

The one double curve consists of the straight line $x = a, y = at, z = at$. Two points P_1 and P_2 on the generatrix g equally distant from M bound a

* Gino Loria, "Spezielle algebraische und transzendente Kurven," 2d ed., Vol. 1, pp. 358-369.

† This is the sixth species of the classification of quartic scrolls by Cayley, Collected Works, Vol. 6, p. 328, and the fifth species of that of Cremona in *Memorie della Accademia delle Scienze dell' Istituto di Bologna*, 2 Ser., Vol. 8, pp. 235-250 (1868).

segment which describes a unifacial band of Moebius.* In a similar manner, and in a more general sense, such a band is described on any surface of the class, and is unit or bifacial according as the order of the surface is odd or even. This surface is also discussed from a function theoretic standpoint by Weyl†.

4. Quintic, $p = 3, q = 2$

$$(y^3 - 3x^2y)z^2 + (2x^4 + 4x^2y^2 + 2y^4 - 2ax^3 + 6axy^2)z + (x^2 + y^2 - a^2)(y^3 - 3x^2y) = 0.$$

C_1 is a triple line of the surface, C_2 is single. There is one double curve of order 3 with the parametric equations

$$x = \frac{a(1 + t^2)}{1 - 3t^2}, \quad y = \frac{at(1 + t^2)}{1 - 3t^2}, \quad z = \frac{a(3t - t^3)}{1 - 3t^2}.$$

The projection of this curve upon the xy -plane has the polar equation

$$\rho = \frac{a}{\cos 3\theta},$$

and the cartesian equation

$$x^3 - 3xy^2 - a(x^2 + y^2) = 0.$$

This quintic has the index *AIII* in Schwarz's classification of quintic ruled surfaces, loc. cit., p. 57.

C. Number of Surfaces of a given Order.

When the order of the surface of the class is given, we may ask the question, how many species of the class are there with the same order? When the order is even, say $2n$, then $p + q = n$, and we may form all possible combinations of relative primes p and q with q odd, to satisfy this condition. For example when the order is 12, $p + q = 6$, and we have the possibilities (1) $p = 5, q = 1$; (2) $p = 2, q = 3$; (3) $p = 1, q = 5$, so that there are three surfaces of order 12. When the order is odd, say $p + q = m$, we may proceed in a similar manner; for example when $m = 13$. As q is now even we have the possibilities (1) $p = 11, q = 2$; (2) $p = 9, q = 4$; (3) $p = 7, q = 6$; (4) $p = 5, q = 8$; (5) $p = 3, q = 10$; (6) $p = 1, q = 12$. Hence there are six surfaces of the class of order 13.

In general, when the order is even, say $2n$ and n is even, then there are at most $n/2$ species of the same order; when n is odd, there at most $(n + 1)/2$. When the order is odd, say m , there are at most $(m - 1)/2$ species of the class with this order.

* Werke, Vol. 2, pp. 484-485 and pp. 519-521.

† "Die Idee der riemannschen Fläche," p. 26.